# NONABELIAN DUALITY AS A SYMMETRY OF AUXILIARY INTERACTION

Evgeny Ivanov, Boris Zupnik

BLTP JINR, Dubna

"Frontiers of Mathematical Physics"

Round Table "France - Italy - Russia",
Dubna, 16 - 18 December 2012

*Based on our works: 1) Nucl. Phys. B 618 (2001) 3; hep-th/0110074; 2) Proceedings of XVI Max Born symposium (2001), hep-th/0202203; 3) Phys. Atom. Nucl., 67 (2004)*

# $O(N)$ symmetry and Abelian gauge fields

We consider the nonlinear interactions of Abelian gauge fields $A^k_m, \quad k = 1, 2, \dots N$ and use the bispinor representation of the field-strengths $F^k_{\alpha\beta} = \frac{1}{8}(\sigma^m\bar{\sigma}^n - \sigma^n\bar{\sigma}^m)_{\alpha\beta}(\partial_m A^k_n - \partial_n A^k_m)$. The Bianchi identities read

$$B^k_{\alpha\dot{\alpha}} = \partial^{\dot{\beta}}_{\alpha}\bar{F}^k_{\dot{\alpha}\dot{\beta}} - \partial^{\beta}_{\dot{\alpha}}F^k_{\alpha\beta} = 0$$

where $\bar{F}^k_{\dot{\alpha}\dot{\beta}} = (F^k_{\alpha\beta})^*$.

The nonlinear Lagrangian of $N$ abelian gauge fields

$$L(F^k, \bar{F}^k) = -\frac{1}{2}[(F^k F^k) + (\bar{F}^k\bar{F}^k)] + L^{int}(F^k, \bar{F}^k)$$

is manifestly invariant under the real $O(N)$ transformation

$$\delta F^k_{\alpha\beta} = \xi^{kl}F^l_{\alpha\beta}, \quad \delta\bar{F}^k_{\dot{\alpha}\dot{\beta}} = \xi^{kl}\bar{F}^k_{\dot{\alpha}\dot{\beta}}, \quad \xi^{kl} = -\xi^{lk}.$$

We use the scalar matrix combinations of the field-strengths in the Lagrangian $L(\varphi, \bar{\varphi})$

# $U(N)$ symmetry and auxiliary fields

Our generalized auxiliary-field representation of the $O(N)$ invariant Lagrangian contains $N$ complex auxiliary fields $V^k_{\alpha\beta}$

$$\mathcal{L} = \frac{1}{2}[(F^k F^k) + (\bar{F}^k \bar{F}^k)] - 2[(F^k V^k) + (\bar{F}^k \bar{V}^k)]$$

$$+(V^k V^k) + (\bar{V}^k \bar{V}^k) + E(\nu, \bar{\nu})$$

where $E$ is the $O(N)$ invariant real interaction of the scalar complex variables

$$\nu^{kl} = (V^k V^l), \quad \bar{\nu}^{kl} = (\bar{V}^k \bar{V}^l).$$

The differential equation for this Lagrangian contains the dual fields

$$P^k_{\alpha\beta}(F, V) = i(F^k - 2V^k)_{\alpha\beta}, \quad \bar{P}^k_{\dot{\alpha}\dot{\beta}}(F, V) = -i(\bar{F}^k - 2\bar{V}^k)_{\dot{\alpha}\dot{\beta}}$$

We define the additional linear transformation with the real parameters $\eta^{kl} = \eta^{lk}$

$$\delta F^k = \eta^{kl} P^l = i\eta^{kl}(F^l - 2V^l), \quad \delta P^k = -\eta^{kl} F^l$$

which form the $U(N)$ Lie algebra together with the $O(N)$ transformations. The corresponding $U(N)$ transformations of the auxiliary fields read

$$\delta V^k = (\xi^{kl} - i\eta^{kl})V^l, \quad \delta \bar{V}^k = (\xi^{kl} + i\eta^{kl})\bar{V}^l.$$

Transformations of the scalar variables $\nu^{kl}, \quad \bar{\nu}^{kl}$ can be presented in the matrix form

$$\delta\nu = [\xi, \nu] - i\{\eta, \nu\}, \quad \delta\bar{\nu} = [\xi, \bar{\nu}] + i\{\eta, \bar{\nu}\}.$$

We define the Hermitian matrix variables

$$a^{kl} = (\bar{\nu}\nu)^{kl}, \quad \bar{a}^{kl} = (\nu\bar{\nu})^{kl}, \quad \delta a = [\xi, a] + i[\eta, a]$$

and use the matrix relations for the matrix polynomials

$$(a^n\bar{\nu})^{kl} = (\bar{\nu}\bar{a}^n)^{kl} = (a^n\bar{\nu})^{lk}, \quad (\nu a^n)^{kl} = (\bar{a}^n\nu)^{kl} = (\nu a^n)^{lk}.$$

The independent real $U(N)$ invariants are $A_n = \frac{1}{n}\mathrm{Tr}\, a^n$ where $n = 1, 2, \dots N$

$$dA_n = \mathrm{Tr}\,(daa^{n-1}), \quad \frac{\partial A_n}{\partial a^{kl}} = (a^{n-1})^{kl}.$$

The basic algebraic equation for the Lagrangian $\mathcal{L}$ is derived by the $V^k$ variation

$$(F^k - V^k)_{\alpha\beta} = \frac{1}{2}(F^k - iP^k)_{\alpha\beta} = \mathcal{E}_{kl} V^l_{\alpha\beta}, \quad \mathcal{E}_{kl} = \frac{\partial \mathcal{E}}{\partial \nu^{kl}}.$$

The scalar matrix algebraic equation follows from this relation

$$\varphi^{kl} = [\delta^{kr} + \mathcal{E}_{kr}]\nu^{rs}[\delta^{sl} + \mathcal{E}_{sl}].$$

We introduce the matrix function $E(a)$ and consider the trace representation of the auxiliary interaction

$$\mathcal{E} = \mathrm{Tr}\, E(a), \quad d\mathcal{E} = \mathrm{Tr}\,(daE_a) = (da^{lk} E^{kl}_a) = dA_n \mathcal{E}_n$$

$$E(a) = \sum_{k=1}^{\infty} e_k a^k, \quad E_a = \sum_{k=1}^{\infty} k e_k a^{k-1}, \quad \mathcal{E}_n = \frac{\partial \mathcal{E}}{\partial A_n}$$

where $e_k$ are some real coefficients. This representation can be used in the basic algebraic equation

$$\frac{\partial \mathcal{E}}{\partial \nu^{kl}} = (E_a)^{kr} \bar{\nu}^{rl} = \mathcal{E}_1 \bar{\nu}^{kl} + \mathcal{E}_2 a^{kr} \bar{\nu}^{rl} + \mathcal{E}_3 a^{kj} a^{jr} \bar{\nu}^{rl} + \ldots$$

The $U(N)$ duality is equivalent to the invariance of the auxiliary interaction $\mathcal{E} = \mathrm{Tr} E(a)$. The $U(N)$ covariant auxiliary equation of motion can be interpreted as the twisted self-duality equation

$$(F^k - V^k)_{\alpha\beta} = (E_a)^{kr} \bar{\nu}^{rl} V^l_{\alpha\beta}.$$

The solution of the twisted self-duality equation has the simple form

$$V^k_{\alpha\beta} = F^l_{\alpha\beta} G_{kl}(\varphi, \bar{\varphi}), \quad P^k_{\alpha\beta} = 2iF^l_{\alpha\beta} L_{kl} = iF^l_{\alpha\beta}[\delta_{kl} - 2G_{kl}]$$

$$G_{kl} = [\delta^{kl} + \mathcal{E}_{kl}]^{-1} = \frac{1}{2}\delta^{kl} - \frac{\partial L}{\partial \varphi^{kl}}.$$

This solution allows us to construct uniquely the self-dual Lagrangian in the standard representation $L(\varphi^{kl}, \bar{\varphi}^{kl})$. The Gailard-Zumino-type formula for the $U(N)$ self-dual Lagrangian has the following explicit form in our formalism:

$$\mathcal{L} = \frac{i}{2}[\bar{P}^k(F, V)\bar{F}^k - P^k(F, V)F^k] + [V^k V^k - (F^k V^k)]$$

$$+[\bar{V}^k \bar{V}^k - (\bar{F}^k \bar{V}^k)] + \mathcal{E}$$

where $V^k V^k - (F^k V^k)$ is the complex bilinear $U(N)$ invariant. The $U(N)$ invariance of the auxiliary interaction $\mathcal{E}$ is equivalent to the $U(N)$ self-duality condition.

The alternative $\mu$ representation for the $O(N)$ Lagrangian uses the matrix variables

$$\mu^{kl} = \frac{\partial \mathcal{E}(a)}{\partial \nu^{kl}} = (E_a)^{kr}\bar{\nu}^{rl}, \quad \bar{\mu}^{kl} = \nu^{kr}(E_a)^{rl}$$

$$b^{kl} = \mu^{ks}\bar{\mu}^{sl} = (E_a\bar{\nu}\nu E_a)^{kl} = (aE_a^2)^{kl}, \quad b^{lk} = \bar{b}^{kl} = \bar{\mu}^{kr}\mu^{rl}.$$

The transformation laws of these matrix variables have the following form:

$$\delta\mu = [\xi, \mu] + i\{\eta, \mu\}, \quad \delta b = [\xi, b] + i[\eta, b]$$

We use the basic relations for these matrix variables

$$(b^n\mu)^{kl} = (\mu\bar{b}^n)^{kl} = (b^n\mu)^{lk},$$

$$(\bar{\mu}b^n)^{kl} = (\bar{b}^n\bar{\mu})^{kl} = (\bar{\mu}b^n)^{lk}$$

and construct the independent $U(N)$ invariants in this representation $B_n = \frac{1}{n}\mathrm{Tr}\, b^n$.

The basic transformation from the $(F, V)$ representation to the $\mu$ representation has the form

$$\mathcal{I}(B_n) = \mathrm{Tr}\, I(b) = \mathcal{E} - \nu^{kl}\mu^{kl} - \bar{\nu}^{kl}\bar{\mu}^{kl} = \mathrm{Tr}\,[E - 2aE_a]$$

$$\nu^{kl} = -\frac{\partial \mathcal{I}}{\partial \mu^{kl}} = -(\bar{\mu} I_b)^{kl} = -(\bar{I}_{\bar{b}}\bar{\mu})^{kl}, \quad d\mathcal{I} = \mathrm{Tr}\,(db I_b) = \mathrm{Tr}\,(d\bar{b}\bar{I}_{\bar{b}})$$

where $I(b)$ and $I_b$ are the matrix functions.
It is instructive to consider the covariant matrix equations

$$I(b) = E(a) - 2aE_a, \quad E(a) = I(b) - 2bI_b,$$

$$E_a = -I_b^{-1}, \quad a = bI_b^2, \quad b = aE_a^2$$

which are analogous to the relations between alternative auxiliary representations in the $U(1)$ theory.
The basic scalar variables can be rewritten in this representation

$$\varphi^{kl} = -(\delta^{kr} + \mu^{kr})\frac{\partial \mathcal{I}}{\partial \mu^{rs}}(\delta^{sl} + \mu^{sl}) = -[\bar{\mu} I_b + bI_b + \bar{\mu} I_b\mu + bI_b\mu]^{kl}.$$

We consider the matrix expansions with the coefficients $i_k$

$$I_b = -2+2i_2b+3i_3b^2+\dots, \quad \bar{\mu}I_b = -2\bar{\mu}+2i_2\bar{\mu}b+3i_3\bar{\mu}b^2+\dots,$$

$$bI_b\mu = -2b\mu + 2i_2b^2\mu + 3i_3b^3\mu + \dots$$

The iterative matrix equation has the following form:

$$\bar{\mu} = \frac{1}{2}\varphi - b - \bar{b} - b\mu + i_2\bar{\mu}b + i_2b^2 + i_2\bar{b}^2 + \frac{3}{2}i_3\bar{\mu}b^2 + i_2b^2\mu$$

$$+\frac{3}{2}i_3b^3 + \frac{3}{2}i_3\bar{b}^3 + \frac{3}{2}i_3b^3\mu + \dots$$

Solving this equations for $\bar{\mu}^{kl}(\varphi,\bar{\varphi})$ we can construct the field derivatives of the Lagrangian

$$[\delta^{kl} + \bar{\mu}^{kl}]^{-1} = \frac{1}{2}\delta^{kl} - \frac{\partial L}{\partial\bar{\varphi}^{kl}}.$$

The combined $(F, V, \mu)$ representation for the $U(N)$ self-dual theories has the form

$$L(V, F, \mu) = \frac{1}{2}[(F^k F^k) + (\bar{F}^k \bar{F}^k)] - 2\,[(V^k \cdot F^k) + (\bar{V}^k \cdot \bar{F}^k)]$$
$$+(V^k V^l)(\delta^{kl} + \mu^{kl}) + \bar{V}^k \bar{V}^l(\delta^{kl} + \bar{\mu}^{kl}) + \mathcal{I}(B_n).$$

Excluding the $V^k$ variables from this Lagrangian

$$V^k_{\alpha\beta} = \left[(1+\mu)^{-1}\right]^{kl} F^l_{\alpha\beta}$$

we obtain the $(F, \mu)$ representation for the $U(N)$ duality

$$\tilde{L}(F^k, \mu^{kl}) = \frac{1}{2}(F^k F^l)[(\mu - 1)(1+\mu)^{-1}]^{kl} + \text{c.c.} + \mathcal{I}(B_n)$$

We can use similarity between the $U(1)$ interaction $I(b)$ and the matrix function $I(b)$ in the $U(N)$ case, although the solution of matrix equations is more difficult. The simple partial $U(N)$ interaction uses the one-parametric function $\mathcal{E}(A_1), \quad A_1 = a^{kk} = \nu^{kl}\bar{\nu}^{lk}$, then

$$\mu^{kl} = \mathcal{E}_1 \bar{\nu}^{kl}, \quad \bar{\mu}^{kl} = \mathcal{E}_1 \nu^{kl}, \quad \nu^{kl} = -\mathcal{I}_1 \bar{\mu}^{kl},$$

$$b^{kl}=\mathcal{E}_1^2 a^{kl},\quad B_1=b^{kk}=\mathcal{E}_1^2 A_1,\quad \mathcal{E}_1=-\mathcal{I}_1^{-1}$$

The transformed interaction depends on the trace $B_1$

$$\mathcal{I}(B_1)=\mathcal{E}(A_1)-2A_1\mathcal{E}_1,\quad \nu^{kl}=-\mathcal{I}_1\bar{\mu}^{kl},\quad a^{kl}=\mathcal{I}_1^2 b^{kl}$$

If $\mathcal{E}=\frac{1}{2}A_1$ then $\mathcal{I}=-2B_1$. The twisted self-duality equation has the polynomial form in this case

$$F^k_{\alpha\beta}=V^l_{\alpha\beta}[\delta^{kl}+\frac{1}{2}(\bar{V}^k\bar{V}^l)].$$

The iterative solution of this equation

$$V^k_{\alpha\beta}=[\delta^{kl}-\frac{1}{2}\bar{\varphi}^{kl}+\frac{1}{4}\bar{\varphi}^{kr}\bar{\varphi}^{rl}+\frac{1}{4}\varphi^{kr}\bar{\varphi}^{rl}$$

$$+\frac{1}{4}\bar{\varphi}^{kr}\varphi^{rl}+\dots]F^l_{\alpha\beta}$$

This solution gives us the perturbative self-dual Lagrangian for the simplest interaction $\mathcal{E} = \frac{1}{2}\text{Tr}\, a$

$$L_{SI} = \text{Tr}\left[-\frac{1}{2}\varphi - \frac{1}{2}\bar{\varphi} + \frac{1}{2}\varphi\bar{\varphi} - \frac{1}{4}\varphi^2\bar{\varphi} - \frac{1}{4}\varphi\bar{\varphi}^2\right]$$

$$+\text{Tr}\left[\frac{1}{8}\varphi^3\bar{\varphi} + \frac{1}{4}\varphi^2\bar{\varphi}^2 + \frac{1}{4}(\varphi\bar{\varphi})^2 + \frac{1}{8}\varphi\bar{\varphi}^3 + \ldots\right]$$

# Manifestly self-dual decomposition of the nonlinear U(1) action with auxiliary fields

We analyze the version of our $U(1)$ Lagrangian using two gauge fields $A^1_m, A^2_m$, auxiliary tensor fields $F_{mn}$ and $V_{mn}$

$$\mathcal{L}(F, V, A^1, A^2) = \frac{1}{4}F^{mn}F_{mn} - F^{mn}V_{mn} + \frac{1}{2}F^{mn}F^1_{mn} - \frac{1}{2}F^{mn}\tilde{F}^2_{mn}$$

$$+\frac{1}{4}F^{1mn}F^1_{mn} - F^{1mn}V_{mn} + \frac{1}{2}V^{mn}V_{mn} + \mathcal{E}(v^2 + w^2)$$

where $F^1_{mn} = \partial_m A^1_n - \partial_n A^1_m$ and $F^2_{mn} = \partial_m A^2_n - \partial_n A^2_m$. Excluding the field $A^2_m$ we obtain the Bianchi identity for $F_{mn}$. Solving this identity via $\tilde{A}_m$:

$$F_{mn} = \partial_m \tilde{A}_n - \partial_n \tilde{A}_m$$

we return to the original formulation of our Lagrangian.

We consider the $O(3)$ decomposition of the Lorentz-covariant field-strengths

$$F^1_{mn}(A) = \partial_m A^1_n - \partial_n A^1_m, \quad \tilde{F}^1_{mn} = \frac{1}{2}\varepsilon_{mnrs}F^{1rs}$$

and auxiliary fields $V_{mn}$

$$F^1_{0k}(A) = E^1_k(A) = \partial_0 A^1_k - \partial_k A^1_0,$$

$$F^1_{kl}(A) = \varepsilon_{klj}B^1_j = \partial_k A^1_l - \partial_l A^1_k,$$

$$\tilde{F}^1_{0k} = B^1_k, \quad \tilde{F}^1_{kl} = -\varepsilon_{klj}E^1_j, \quad V_{0k} = V_k, \quad V_{kl} = \varepsilon_{klj}U_j$$

where $k, l, j = 1, 2, 3$.

The 3D decompositions of the covariant scalar combinations of auxiliary fields read

$$\nu = v + iw = \frac{1}{2}U_k U_k - \frac{1}{2}V_k V_k - iV_k U_k,$$

$$a(V, U) = v^2 + w^2 = \frac{1}{4}(U_i U_i)^2 + \frac{1}{4}(V_i V_i)^2$$
$$-\frac{1}{2}(U_i U_i)(V_k V_k) + (V_i U_i)^2$$

By the analogy with the method of A. Tseytlin in the *BI* theory we can preserve two gauge fields

$$A^1_m = (A^1_0, A^1_k), \quad A^2_m = (A^2_0, A^2_k)$$

and use the non-covariant gauge for the auxiliary field $F_{mn} = (F_{0k}, F_{kl})$

$$F_{0k} = F_k, \quad F_{kl} = 0, \quad \tilde{F}_{0k} = 0, \quad \tilde{F}_{kl} = -\varepsilon_{klj} F_{0j} = -\varepsilon_{klj} F_j.$$

The non-covariant form of our bilinear Lagrangian reads

$$L_2(F, V, U, A^1, A^2) = -\frac{1}{2} F_k F_k + 2 F_k V_k - F_k E_k^1 - \frac{1}{2} E_k^1 E_k^1$$

$$+\frac{1}{2} B_k^1 B_k^1 + 2 E_k^1 V_k - 2 B_k^1 U_k + F_k B_k^2 - V_k V_k + U_k U_k.$$

Using the following $O(2)$ transformations

$$\delta F_k = 2\omega U_k - \omega E_k^2 - \omega B_k^1, \quad \delta V_k = \omega U_k, \quad \delta U_k = -\omega V_k,$$

$$\delta E_k^a = \omega \varepsilon^{ab} E_k^b, \quad \delta B_k^a = \omega \varepsilon^{ab} B_k^b$$

we prove the $O(2)$ invariance of the non-covariant bilinear action

$$\delta S_2 = \int d^4x \delta L_2 = \omega \int d^4x (E^1_k B^1_k - E^2_k B^2_k) = 0,$$

$$F^{1mn}\tilde{F}^1_{mn} = -4E^1_k B^1_k = \text{div}, \quad F^{2mn}\tilde{F}^2_{mn} = -4E^2_k B^2_k$$

Now we can exclude the auxiliary field $F_k$ using its algebraic equation

$$F_k = 2V_k - E^1_k + B^2_k.$$

The $O(2)$ invariant Lagrangian with two gauge fields and auxiliary fields $V_k, U_k$ has the form

$$\tilde{L}(A^1, A^2, V, U) = \frac{1}{2}B_k^1 B_k^1 + \frac{1}{2}B_k^2 B_k^2 - E_k^1 B_k^2 + 2V_k B_k^2 - 2B_k^1 U_k$$

$$+V_k V_k + U_k U_k + \mathcal{E}[a(V, U)]$$

where $\mathcal{E}(a) = \frac{1}{2}a + e_2 a^2 + \ldots$ is the invariant auxiliary interaction.

It is evident that

$$\delta\tilde{L}(A^1, A^2, V, U) = -\delta(E_k^1 B_k^2) = \omega(E_k^1 B_k^1 - E_k^2 B_k^2) = \text{div}$$

and other terms are manifestly invariant.

The equations for $V_k$ and $U_k$ can be solved in the lowest orders of the perturbation theory via the fields $B^1_k$ and $B^2_k$

$$V_k = -B^2_k + \frac{1}{4}B^2_k(B^2_l B^2_l) - \frac{1}{4}B^2_k(B^1_l B^1_l) + \frac{1}{2}B^1_k(B^2_l B^1_l) + O(B^5),$$

$$U_k = B^1_k - \frac{1}{4}B^1_k(B^1_l B^1_l) + \frac{1}{4}B^1_k(B^2_l B^2_l) - \frac{1}{2}B^2_k(B^2_l B^1_l) + O(B^5),$$

then we obtain the manifestly self-dual Lagrangian depending on these fields

$$L(A^1, A^2) = -\frac{1}{2}B^1_k B^1_k - \frac{1}{2}B^2_k B^2_k - E^1_k B^2_k + \frac{1}{8}(B^1_i B^1_i)^2 + \frac{1}{8}(B^2_i B^2_i)^2$$

$$-\frac{1}{4}(B^1_i B^1_i)(B^2_k B^2_k) + \frac{1}{2}(B^1_i B^2_i)^2 + O(B^6)$$

The covariant auxiliary interaction $\mathcal{E}(a)$ generates the non-polynomial self-dual Lagrangians $L(A^1, A^2)$. Note that all terms of the fixed order in this representation are invariant under the linear $O(2)$ transformation.

Our Lorentz-covariant self-dual Lagrangian with the arbitrary invariant auxiliary interaction is equivalent to the non-covariant and manifestly O(2) invariant Lagrangian. The similar non-covariant and manifestly U(N) invariant Lagrangian with $N$ pair of the gauge fields $A^1$ and $A^2$ can be constructed

THANK YOU